GOSFORD HIGH SCHOOL

2010 HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 2

ASSESSMENT TASK 2

PART A

Time Allowed – 60 minutes

+ 5 minutes reading time

All necessary working should be shown.

Full marks may not be awarded for unnecessarily untidy work or work that is poorly organized.

Students must begin each new question on a new page.

Questions will be collected separately at the conclusion of the assessment task.

All questions are to be attempted.

GRAPHS (21 marks)

a) (i) Sketch $y = x^3 + 1$ (1)

(ii) Hence, on the **same** graph, sketch $y = \sqrt{x^3 + 1}$ (2)

b) (i) Sketch $y = 2^x - 1$ (1)

(ii) Hence, on the **same** graph, sketch $y = \dfrac{1}{2^x - 1}$ (2)

c) (i) Sketch $y = 1 - |x|$ (1)

(ii) Hence, on a **new** graph, sketch $|y| = 1 - |x|$ (2)

d) $f(x) = 2\cos x$ where $-\pi \leq x \leq \pi$.
Sketch $y = f(x)$ and $y = [f(x)]^3$ on the **same** number plane. (4)

e) Sketch $y = x\sin x$ for $-2\pi \leq x \leq 2\pi$ (4)

f) Sketch the curve $y = \dfrac{x^2 + 3x}{x + 1}$, showing all intercepts with the co-ordinate axes and asymptotes. (4)

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HSC Assessment Task 2 — Year 12 2010 — Mathematics Extension 2

COMPLEX NUMBERS (21 marks)

a) Let $z = 3+2i$ and $w = 1-5i$.

Find, in the form $x+iy$.

(i) zw (ii) $\dfrac{w}{\bar{z}}$ (2)

b) (i) Express $-1+i$ and $1+i\sqrt{3}$ in modulus-argument form. (2)

(ii) Hence express $\dfrac{(-1+i)^4}{(1+i\sqrt{3})^5}$ in the form $x+iy$. (3)

c) Sketch the following locus on separate Argand diagrams

(i) $-1 \le \text{Im}(z) < 2$ (2)

(ii) $\arg(z-i) = \arg(z+1)$ (2)

d) (i) Sketch $\left|z-1-i\sqrt{3}\right| = 2$, (2)

(ii) Hence find the range of values of Arg(z) (2)

e) Describe in simple geometric terms the locus of z if $\frac{1}{2}(z+\bar{z}) = |z|-2$ (3)

f) Given $z = \dfrac{1-\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}$, show that z is purely imaginary (3)

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

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2010 HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 2

ASSESSMENT TASK 2

PART B

Time Allowed – 60 minutes

+ 5 minutes reading time

All necessary working should be shown.

Full marks may not be awarded for unnecessarily untidy work or work that is poorly organized.

Students must begin each new question on a new page.

Questions will be collected separately at the conclusion of the assessment task.

All questions are to be attempted.

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HSC Assessment Task 2 Year 12 2010 Mathematics Extension 2

INTEGRATION (20 marks)

a) Find

(i) $\int \frac{e^{\tan x}}{\cos^2 x} dx$ (1)

(ii) $\int \tan^4 x \ dx$ (3)

(iii) $\int x \cos(2x) dx$ (2)

(iv) $\int \frac{dx}{\sqrt{x^2 - 8x + 25}}$ (2)

b) Evaluate

(i) $\int_0^1 \sqrt{4 - x^2} dx$ using the substitution $x = 2\sin\theta$ (4)

(ii) $\int_2^{10} \frac{x^2 dx}{\sqrt{x-1}}$ (4)

c) (i) Show that $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$, (2)

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \sin(5x)\cos(3x) dx$ (2)

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POLYNOMIALS (22 marks)

a) Factorise $x^4 - x^2 - 20$ over the

(i) Real field (R) (1)

(ii) Complex field (C). (1)

b) If $2 + i$ is a root of the polynomial $S(x) = x^3 - 2x^2 - 3x + 10$, determine the other two roots. (2)

c) Find the roots of the polynomial $x^4 - 6x^3 + 12x^2 - 10x + 3$ given that it has a root of multiplicity 3 (3)

d) The equation $x^3 - 5x^2 + 8x + 24 = 0$ has roots α , β and γ.

Find the polynomial equation with roots $\alpha^2,\ \beta^2\ \ and\ \ \gamma^2$ (3)

e) Resolve $\dfrac{2x^2 + x + 5}{(x-3)(x^2+4)}$ into partial fractions over the Real field. (4)

f) Find all the zeros of the polynomial $\phi(x) = 2x^3 - 5x^2 + x + 3$, given that it has one rational zero. (3)

g) (i) If $z = \cos 6 + i \sin 6$, show that

$z + z^{-1} = 2\cos 6$ and that $z^n + z^{-n} = 2\cos(n\theta)$ (2)

(ii) By expanding $\left(z + \dfrac{1}{z}\right)^4$, find an expression for $\cos^4 6$ in the form

$A\cos(4\theta) + B\cos(2\theta) + C$ (3)

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

COMPLEX NUMBERS

a) (i) $zw = (3+2i)(1-5i)$

$= 3 - 15i + 2i + 10$

$= 13 - 13i$

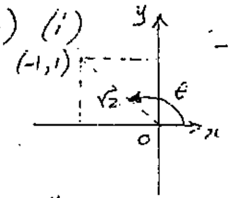
(ii) $\dfrac{w}{\bar{z}} = \dfrac{1-5i}{3-2i} \times \dfrac{3+2i}{3+2i}$

$= \dfrac{13-13i}{9+4}$

$= \dfrac{13-13i}{13}$

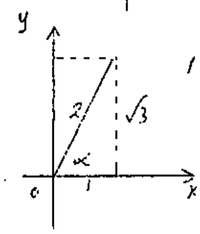
$= 1 - i$

b) (i)



$-1+i = \sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$



$1+i\sqrt{3} = 2\,\text{cis}\,\frac{\pi}{3}$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(ii) $\dfrac{(-1+i)^4}{(1+i\sqrt{3})^5} = \dfrac{\left(\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}\right)^4}{\left(2\,\text{cis}\,\frac{\pi}{3}\right)^5}$

$= \dfrac{4\,\text{cis}\,3\pi}{32\,\text{cis}\,\frac{5\pi}{3}}$

$= \frac{1}{8}\,\text{cis}\left(\frac{4\pi}{3}\right)$

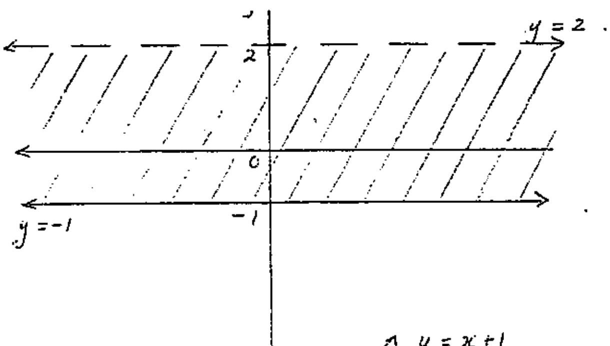
$= \frac{1}{8}\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

$= \frac{1}{8}\left[-\frac{1}{2} - i \times \frac{\sqrt{3}}{2}\right]$

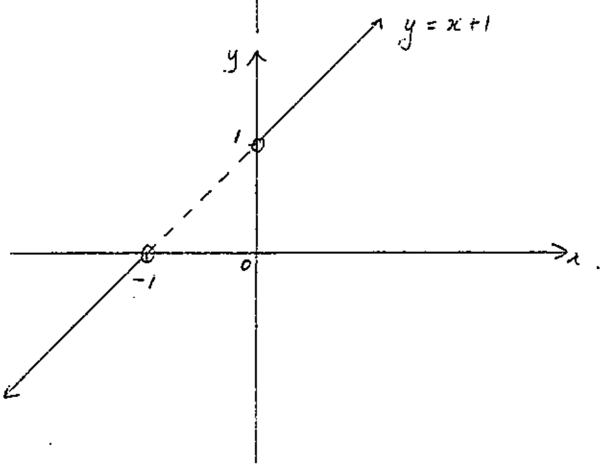
$= -\frac{1}{16}\left(1 + i\sqrt{3}\right)$

$= -\frac{1}{16} - \frac{i\sqrt{3}}{16}$

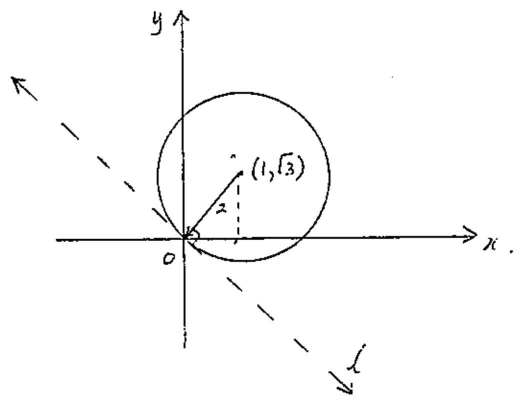
c) (i)



(ii)



d) (i)



(ii) Gradient of line $(l) = -\frac{1}{\sqrt{3}}$

$\therefore -\frac{\pi}{6} < \text{Arg}(z) < \frac{5\pi}{6}$

e) $z = x + iy$

$\therefore \frac{1}{2}(2x) = \sqrt{x^2+y^2} - 2$

$x + 2 = \sqrt{x^2+y^2}$

$x^2 + 4x + 4 = x^2 + y^2$

$y^2 = 4x + 4$

Parabola with vertex $(-1, 0)$ and

f) $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$.

$1 + \cos 2\theta = 2\cos^2\theta$ $\qquad$ $2\sin^2\theta = 1 - \cos 2\theta$

$$z = \frac{2\sin^2\theta + 2i\sin\theta\cos\theta}{2\cos^2\theta - 2i\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta(\sin\theta + i\cos\theta)}{2\cos\theta(\cos\theta - i\sin\theta)}$$

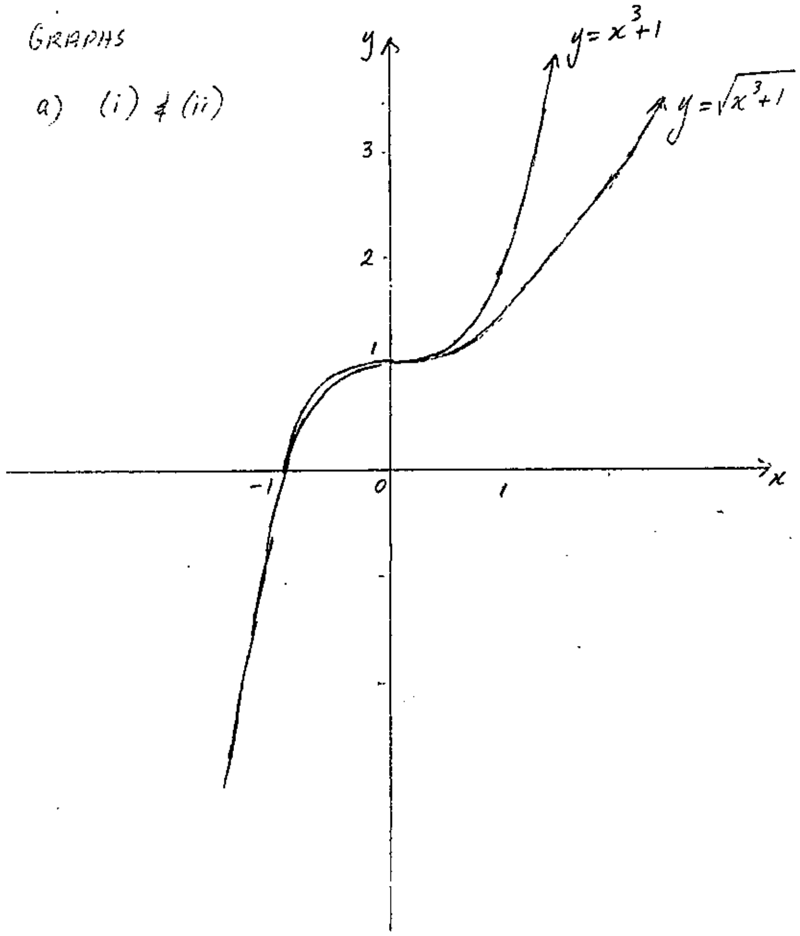
$$= \frac{2i\sin\theta(\cos\theta - i\sin\theta)}{2\cos\theta(\cos\theta - i\sin\theta)}$$

$$= i\tan\theta.$$

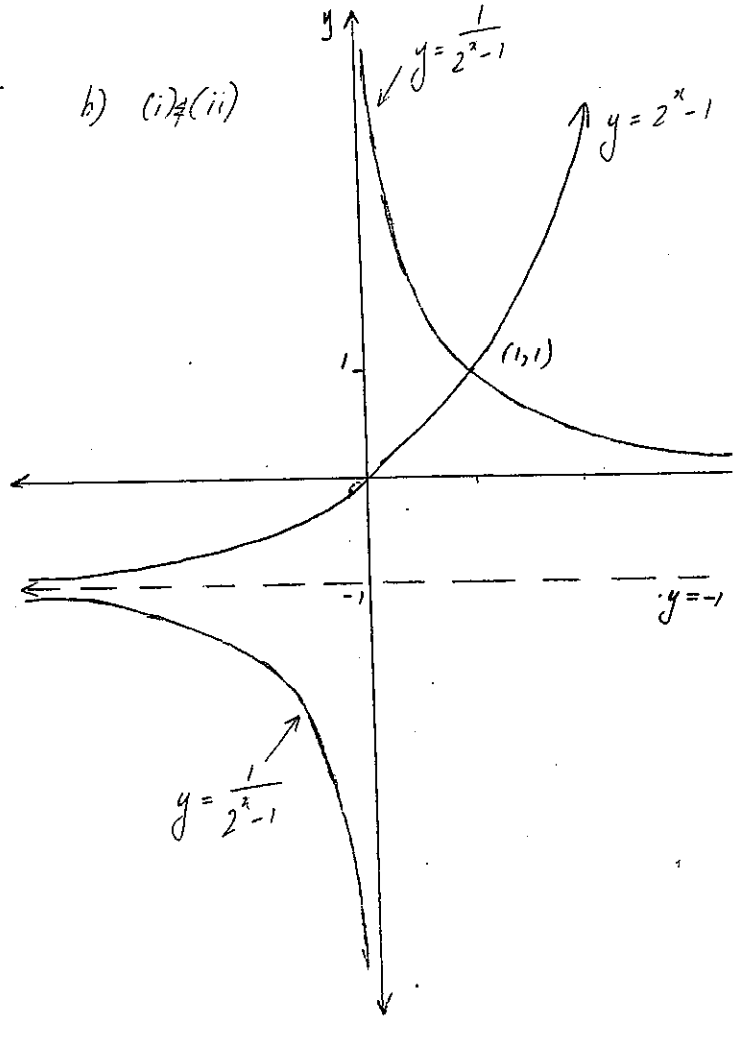
$\therefore z$ is purely imaginary.

GRAPHS

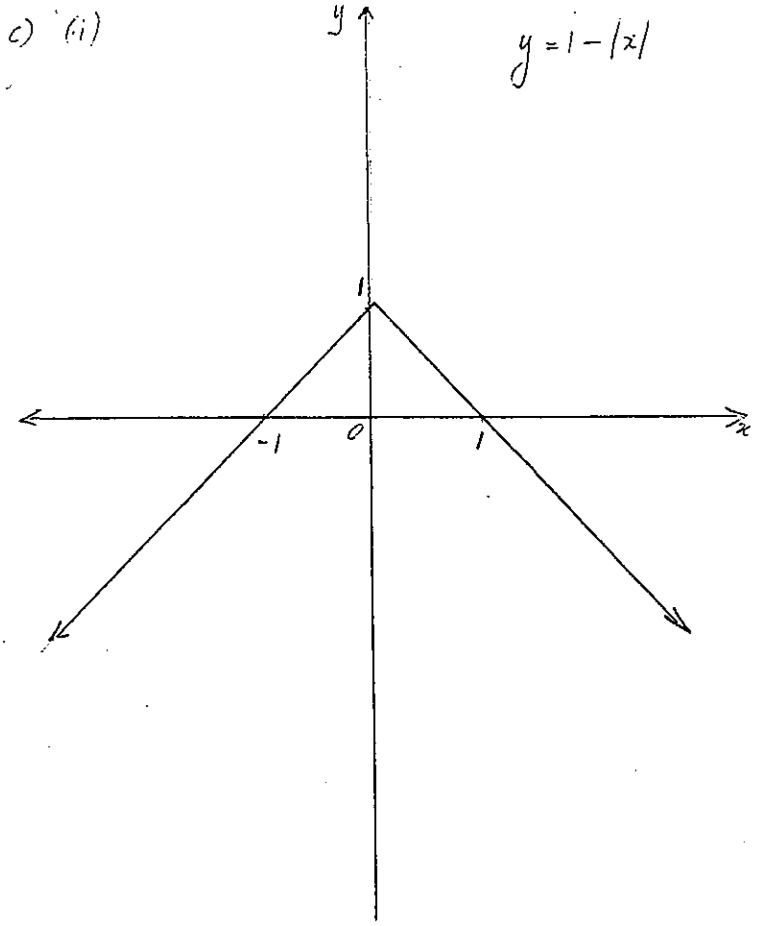
a) (i) & (ii)



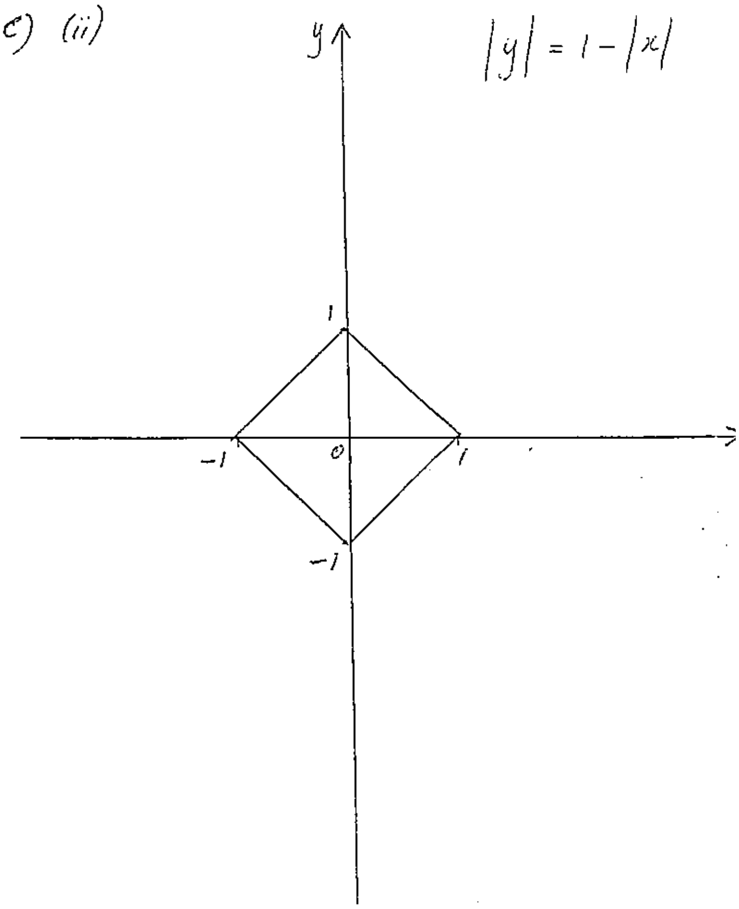
b) (i)&(ii)



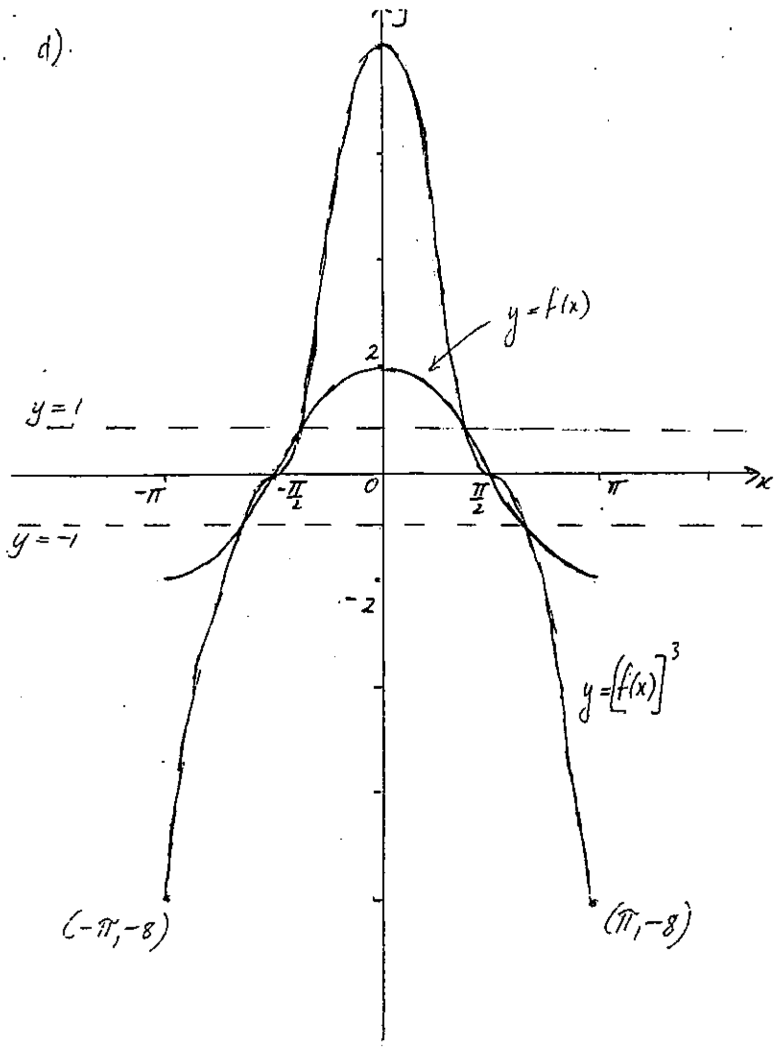
c) (i)


y
$y = 1 - |x|$
1
-1
0
1
x

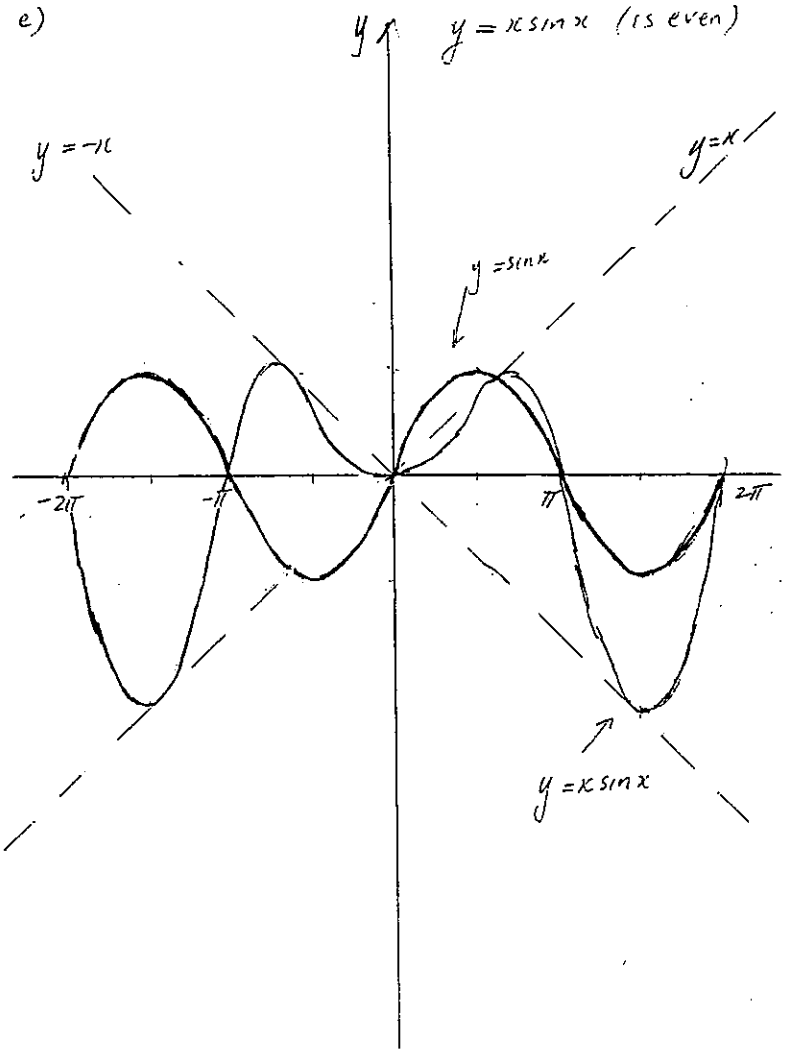
c) (ii)


y
$|y| = 1 - |x|$
1
-1
0
1
-1

d)

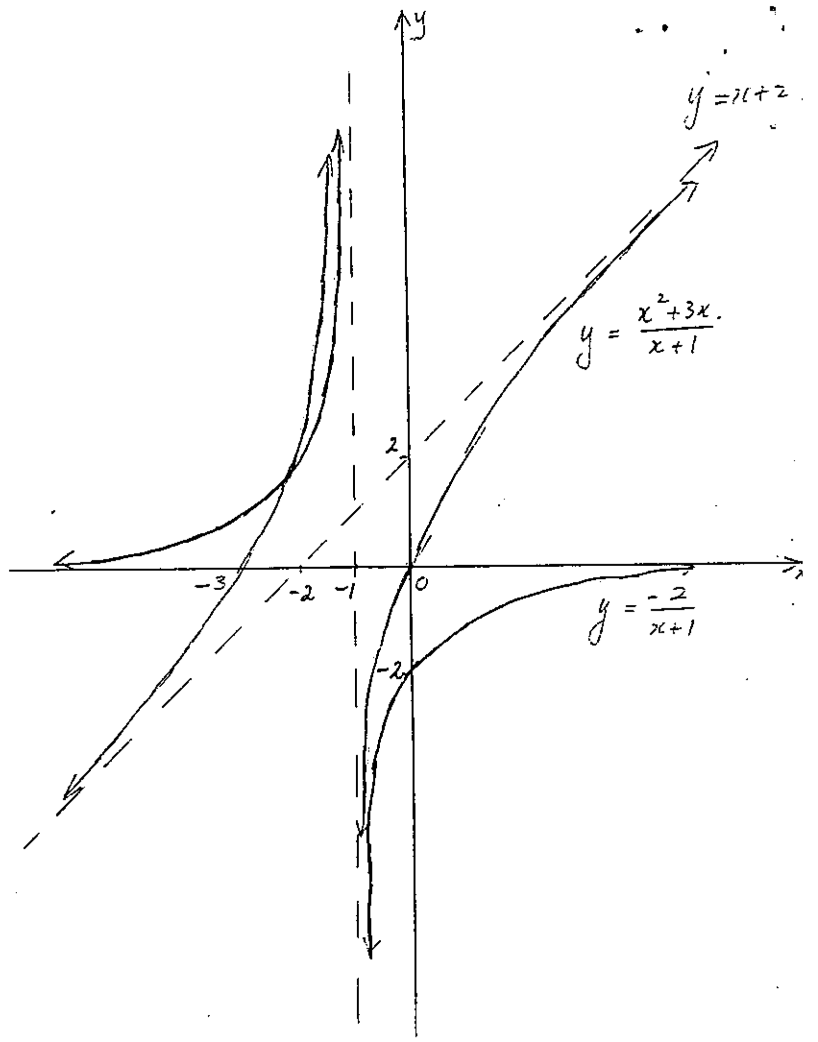

y
2
$y = f(x)$
$y = 1$
$y = -1$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
x
-2
$y = [f(x)]^3$
$(-\pi, -8)$
$(\pi, -8)$

e)


y
$y = x \sin x$ (is even)
$y = -x$
$y = x$
$y = \sin x$
-2π
$-\pi$
π
2π
$y = x \sin x$

f) $\dfrac{x^2+3x}{x+1} = x+2 + \dfrac{-2}{x+1}$

$$\begin{array}{r} x+2 \\ x+1 \overline{)\, x^2+3x} \\ \underline{x^2+x} \\ 2x \\ \underline{2x+2} \\ -2 \end{array}$$



INTEGRATION

a) (i) $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int \sec^2 x \cdot e^{\tan x}\,dx$

$= e^{\tan x} + c$

(ii) $\displaystyle\int \tan^4 x\,dx = \int \tan^2 x \cdot \tan^2 x\,dx$

$\displaystyle = \int (\sec^2 x - 1)\cdot \tan^2 x\,dx$

$\displaystyle = \int (\sec^2 x \tan^2 x - \tan^2 x)\,dx$

$\displaystyle = \int \left[\sec^2 x \tan^2 x - (\sec^2 x - 1)\right]dx$

$\displaystyle = \int (\sec^2 x \tan^2 x - \sec^2 x + 1)\,dx$

$= \dfrac{\tan^3 x}{3} - \tan x + x + c$

(iii) Let $u = x$ and $\dfrac{dv}{dx} = \cos 2x$

$\dfrac{du}{dx} = 1$ $\qquad V = \frac{1}{2}\sin 2x$

$\displaystyle\int x\cdot\cos(2x)\,dx = \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx$

$= \frac{x}{2}\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + c$

$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + c$

(iv) $\displaystyle\int \frac{dx}{\sqrt{x^2-8x+25}} = \int \frac{dx}{\sqrt{x^2-8x+16+9}}$

$\displaystyle = \int \frac{dx}{\sqrt{(x-4)^2+9}}$

Let $u = x-4$

$\dfrac{du}{dx} = 1 \longrightarrow \dfrac{dx}{du} = 1$

$\displaystyle\therefore \int \frac{dx}{\sqrt{x^2-8x+25}} = \int \frac{1}{\sqrt{u^2+9}}\cdot 1\,du$

(iv) Continued

$$\int \frac{dx}{\sqrt{x^2-8x+25}} = \log_e\left[u + \sqrt{u^2+9}\right] + c$$

$$= \log_e\left[(x-4) + \sqrt{(x-4)^2+9}\right] + c$$

$$= \log_e\left[(x-4) + \sqrt{x^2-8x+25}\right] + c$$

b) (i)

$x = 2\sin\theta$. when $x = 1$, $\theta = \frac{\pi}{6}$

$\frac{dx}{d\theta} = 2\cos\theta$ $\quad x = 0$, $\theta = 0$

$$\therefore \int_0^1 \sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta.$$

$$= \int_0^{\frac{\pi}{6}} 2\cos\theta\cdot 2\cos\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{6}} \cos^2\theta\,d\theta.$$

Now $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\therefore \int_0^1 \sqrt{4-x^2}\,dx = 2\int_0^{\frac{\pi}{6}} (1+\cos 2\theta)\,d\theta.$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= 2\left[\frac{\pi}{6} + \frac{1}{2}\times\frac{\sqrt{3}}{2} - 0\right]$$

$$= 2\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right]$$

(ii) Let $u^2 = x-1$ $\quad$ when $x = 10$, $u = 3$

$x = u^2+1$ $\quad x = 2$, $u = 1$

$\frac{dx}{du} = 2u$

$$\therefore \int_2^{10} \frac{x^2\,dx}{\sqrt{x-1}} = \int_1^3 \frac{(u^2+1)^2}{u}\cdot 2u\,du.$$

$$= 2\int_1^3 (u^4+2u^2+1)\,du.$$

$$= 2\left[\frac{u^5}{5} + \frac{2u^3}{3} + u\right]_1^3$$

b) (ii) (continued)

$$\int_2^{10} \frac{x^2\,dx}{\sqrt{x-1}} = 2\left[3 + 18 + \frac{243}{5} - 1 - \frac{2}{3} - \frac{1}{5}\right]$$

$$= 2\left[\frac{242}{5} + \frac{58}{3}\right]$$

$$= \frac{2032}{15}$$

c) (i)

$$R.H.S. = \frac{1}{2}\left[\sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B\right]$$

$$= \frac{1}{2}\left[2\sin A\cos B\right]$$

$$= \sin A\cos B$$

$$= L.H.S.$$

(ii)
$$\int_0^{\frac{\pi}{2}} \sin 5x\cos 3x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 8x + \sin 2x)\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{16}\left[\cos 8x + 4\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin 5x\cos 3x\,dx = -\frac{1}{16}\left[1 - 4 - (1+4)\right]$$

$$= -\frac{1}{16}\left[-8\right]$$

$$= \frac{1}{2}$$

POLYNOMIALS

a) (i) $x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$

$= (x - \sqrt{5})(x + \sqrt{5})(x^2 + 4)$

(ii) $x^4 - x^2 - 20 = (x - \sqrt{5})(x + \sqrt{5})(x + 2i)(x - 2i)$

b) $S(2+i) = (2+i)^3 - 2(2+i)^2 - 3(2+i) + 10$

$= (2+i)(4 - 1 + 4i) - 2(3 + 4i) - 6 - 3i + 10$

$= (2+i)(3+4i) - 6 - 8i - 6 - 3i + 10$

$= 6 + 8i + 3i - 4 - 6 - 8i - 6 - 3i + 10$

$= 0$

$\therefore$ $(2+i)$ is a zero.

$\therefore$ $2 - i$ is a zero as zeros of a Real Polynomial appear in conjugate pairs.

Sum of Roots $= \dfrac{-b}{a}$.

$(2+i) + (2-i) + \alpha = 2$

$\alpha = -2$

$\therefore$ Roots are $(2+i)$, $(2-i)$ & -2

c) Let $P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$

$P'(x) = 4x^3 - 18x^2 + 24x - 10$

$P''(x) = 12x^2 - 36x + 24$

For a root of multiplicity of 3

$P''(x) = P'(x) = P(x) = 0$

$\therefore$ $12x^2 - 36x + 24 = 0$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2, 1$

$P'(1) = 4 - 18 + 24 - 10$

$= 0$

$P(1) = 1 - 6 + 12 - 10 + 3$

$\therefore P(1) = 0 = P'(1) = P''(1)$.

$\therefore$ $x = 1$ is (only) root of multiplicity 3

i.e. $(x-1)^3$ is a factor.

$\therefore$ $P(x) = (x-1)^3(x-3)$ By Inspection.

$\therefore$ Roots of $P(x)$ are 1, 3

d) α is a root.

$\therefore$ $\alpha^3 - 5\alpha^2 + 8\alpha + 24 = 0$

We want $x = \alpha^2$ to be a root.

$\therefore$ $\alpha = \pm\sqrt{x}$

$\therefore (\pm\sqrt{x})^3 - 5(\pm\sqrt{x})^2 + 8(\pm\sqrt{x}) + 24 = 0$

$(\pm\sqrt{x})^3 + 8(\pm\sqrt{x}) = 5x - 24$

$\pm\sqrt{x}\,[x + 8] = 5x - 24$.

Squaring both sides

$x(x+8)^2 = (5x - 24)^2$

$x(x^2 + 16x + 64) = 25x^2 - 240x + 576$

d) (continued)

$x^3 + 16x^2 + 64x = 25x^2 - 240x + 576$

$\therefore$ $x^3 - 9x^2 - 304x - 576 = 0$

e) Let $\dfrac{2x^2 + x + 5}{(x-3)(x^2+4)} = \dfrac{a}{x-3} + \dfrac{bx + c}{x^2 + 4}$

$\therefore$ $2x^2 + x + 5 \equiv ax^2 + 4a + (x-3)(bx + c)$

$= ax^2 + 4a + bx^2 + cx - 3bx - 3c$

$= (a+b)x^2 + (c - 3b) + 4a - 3c$

$\therefore$ Equating Coefficients

$a + b = 2$; $c - 3b = 1$; $4a - 3c =$

$3a + 3b = 6$ +

$c - 3b = 1$

$3a + c = 7 \longrightarrow 9a + 3c = 21$

$4a - 3c = 5$

$13a = 26$

$a = 2$

$\therefore$ $b = 0$ and $c = 1$

e) (continued)

$$\frac{2x^2+x+5}{(x-3)(x^2+4)} = \frac{2}{x-3} + \frac{1}{x^2+4}$$

f) Possible rational zeros are

$\pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}$

$\phi(\frac{1}{2}) \neq 0$, $\phi(-\frac{1}{2}) \neq 0$

$$\phi\left(\frac{3}{2}\right) = \frac{27}{4} - \frac{45}{4} + \frac{6}{4} + \frac{12}{4} = 0$$

$\therefore$ $x = \frac{3}{2}$ is a root

$(2x-3)$ is a factor of $\phi(x)$

$$\therefore \phi(x) = (2x-3)(x^2+ax-1)$$
$$= 2x^3+2ax^2-2x-3x^2-3ax+3$$
$$= 2x^3+(2a-3)x^2-(2+3a)x+3$$

$\therefore$ $2a-3=-5$ Equating coefficients

$a=-1.$

$$\therefore \phi(x) = (2x-3)(x^2-x-1)$$

$\therefore$ Zeros are $\frac{3}{2}$, $\frac{1\pm\sqrt{5}}{2}$

g)(i) $z = \cos w + i \sin w$

$$z^{-1} = (\cos w + i\sin w)^{-1}$$
$$= \cos(-w) + i\sin(-w)$$
$$z^{-1} = \cos w - i\sin w$$

$\therefore$ $z + z^{-1} = 2\cos w$

$$z^n = (\cos w + i\sin w)^n$$
$$= \cos(nw) + i\sin(nw)$$
$$z^{-n} = \cos(-nw) + i\sin(-nw)$$
$$= \cos(nw) - i\sin(nw)$$

$\therefore$ $z^n + z^{-n} = 2\cos(nw)$

(ii) $$\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$\therefore (2\cos w)^4 = z^4 + \frac{1}{z^4} + 4\left(z^2+\frac{1}{z^2}\right) + 6$$

$$= 2\cos(4w) + 4\times 2\cos(2w) + 6$$

$$16\cos^4 w = 2\cos(4w) + 8\cos(2w) + 6$$

$$\cos^4 w = \frac{1}{8}\cos(4w) + \frac{1}{2}\cos(2w) + \frac{3}{8}$$